ISRAEL JOURNAL OF MATHEMATICS 113 (1999), 205–230

BETWEEN P-POINTS AND NOWHERE DENSE ULTRAFILTERS

BY

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ABSTRACT

We discuss the problem of existence and that of generic existence for various classes of ultrafilters on ω. For example, we prove it is consistent that there are nowhere dense ultrafilters while there are no measure zero ultrafilters, and that there are measure zero ultrafilters while there are no P-points. We also prove that every filter base of size $< \mathfrak{c}$ can be extended to a nowhere dense ultrafilter iff $\mathrm{cof}(\mathcal{M}) = \mathfrak{c}$, and that every filter base of size $< \mathfrak{c}$ can be extended to an ordinal ultrafilter iff $\mathfrak{d} = \mathfrak{c}$. Along the way we get a few new results on cardinal invariants of the continuum.

Introduction

This work deals with the problem of existence and that of generic existence for various kinds of free ultrafilters on the natural numbers ω. The classes of ultrafilters we are interested in can all be defined in the following general framework of Baumgartner [B].

Let $\mathcal{I}$ be a family of subsets of a set X which contains all singletons and is closed under subsets. An ultrafilter $\mathcal{U}$ on ω is called an **$\mathcal{I}$-ultrafilter** if for all functions $f\colon \omega \to X$ there is $A \in \mathcal{U}$ such that $f[A] \in \mathcal{I}$. Clearly, if $\mathcal{I} \subseteq \mathcal{J}$, then any $\mathcal{I}$-ultrafilter is also a $\mathcal{J}$-ultrafilter. Also, if $\langle \mathcal{I} \rangle$ is the ideal generated by $\mathcal{I}$, then $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter iff it is an $\langle \mathcal{I} \rangle$-ultrafilter so that it suffices to consider the case where $\mathcal{I}$ is an ideal. However, for technical purposes, we shall sometimes deal with arbitrary families $\mathcal{I}$. To describe a few examples (most of which are

Received October 14, 1997

due to Baumgartner [B]) first let X be the reals 2^ω. If $\mathcal{I}$ consists of all finite sets and all converging sequences, then the $\mathcal{I}$-ultrafilters simply coincide with the well-known class of **P-points** (see [B, Lemma 1.3]). Other interesting examples arise when choosing for $\mathcal{I}$ a larger family of subsets of 2^ω which are still small in a topological sense, e.g. $\mathcal{I}$ is $\mathcal{C}$, the sets with countable closure, or $\mathcal{E}_0$, the sets A with closure $\bar{A}$ of measure zero, or $\mathcal{M}_0$, the nowhere dense sets, or the discrete sets, or the scattered sets. We refer to the ultrafilters corresponding to those families as **countable closed**, **measure zero**, **nowhere dense**, **discrete** and **scattered ultrafilters**, respectively. All of these ultrafilter classes contain the P-points, the first three are ordered by inclusion, both the countable closed and the discrete ultrafilters are scattered, and the scattered ones are in turn nowhere dense. See Figure 1 below.

A natural question to ask about these classes is whether they can be shown to be non-empty in ZFC — or, more generally, whether it is consistent that there are ultrafilters of one kind while there are none of another kind. Many years ago, Shelah proved there may be no P-points ([Sh], see also [BJ, 4.4.B]). Recently he established that it is even consistent there are no nowhere dense ultrafilters [Sh594]. He also showed that nowhere dense ultrafilters may exist while there are no P-points [B, Theorem 3.1]. We refine the latter result to

THEOREM A: *It is consistent that there are nowhere dense ultrafilters, yet there are no measure zero ultrafilters.*

THEOREM B: *It is consistent that there are measure zero ultrafilters, yet there are no countable closed ultrafilters (a fortiori, there are no P-points).*

The proofs of these results are given in section 3 of the present paper after some preliminary work in the preceding sections which we shall explain below.

Different examples of $\mathcal{I}$-ultrafilters arise as follows. Let $X = \omega_1$, fix $\alpha < \omega_1$ and let $\mathcal{I} = \mathcal{J}^{\omega_1}_\alpha$ be the subsets of ω_1 of order type $< \alpha$. $\mathcal{I}$ is an ideal iff α is an indecomposable ordinal. $\mathcal{U}$ is called an **α-ultrafilter** iff it is a $\mathcal{J}^{\omega_1}_{\alpha\cdot\omega}$-ultrafilter (iff, given any $f\colon \omega \to \omega_1$, there is $U \in \mathcal{U}$ with o.t.$f[U] \leq \alpha$). An ultrafilter which is an α-ultrafilter for some $\alpha < \omega_1$ is referred to as an **ordinal ultrafilter**. This class of ultrafilters was also introduced by Baumgartner [B, section 4] and further studied by Laflamme [La]. The ω-ultrafilters ($\mathcal{J}^{\omega_1}_{\omega^2}$-ultrafilters) coincide with the P-points [B, Theorem 4.1] so that every P-point is an ordinal ultrafilter. The inclusion relations between the ultrafilter classes are summarized in Figure 1.

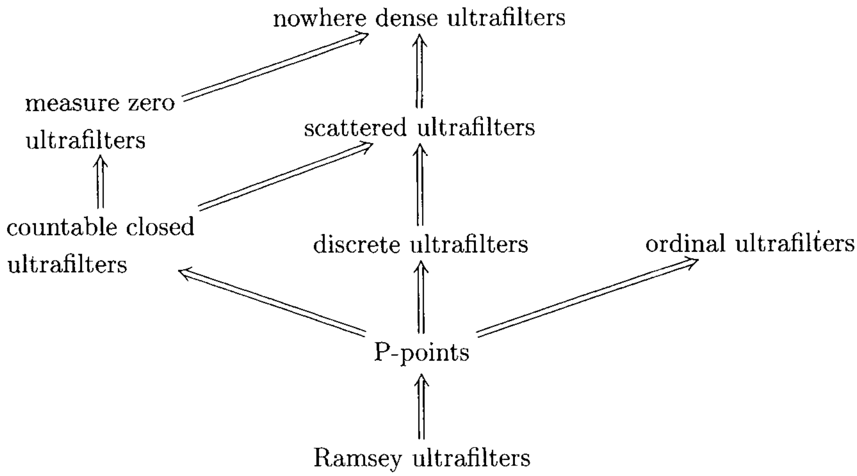

Figure 1. Inclusion relations between the ultrafilter classes.

The answer to the following is not known.

CONJECTURE (Baumgartner [B]): *It is consistent there are no ordinal ultrafilters.*

However, Theorem E below gives us strong evidence that this is the case.

To get a better understanding of both this problem and the techniques needed for the proofs of Theorems A and B, the following notion is crucial. Let $\mathcal{K}$ be a class of ultrafilters. Say ultrafilters from $\mathcal{K}$ **exist generically** iff each filter base of size less than $\mathfrak{c}$, the cardinality of the continuum, can be extended to an ultrafilter belonging to $\mathcal{K}$. This notion is stronger than mere existence. One of the nice things about it is that, unlike the latter, generic existence can usually be characterized in terms of a simple statement about cardinal invariants of the continuum. The oldest result of this kind is a theorem of Ketonen ([Ke], see also [BJ, Theorem 4.4.5]) saying that P-points exist generically iff $\mathfrak{d} = \mathfrak{c}$. Canjar (and independently, but later, Bartoszyński and Judah; see [Ca] and [BJ, Theorem 4.5.6]) proved that generic existence of Ramsey ultrafilters is equivalent to $\text{cov}(\mathcal{M}) = \mathfrak{c}$. Recently, Eisworth [E1] showed the equivalence of the latter with generic existence of stable ordered-union ultrafilters. We show in section 2 of the present work:

THEOREM C: *The following are equivalent:*

(a) $\text{cof}(\mathcal{M}) = \mathfrak{c}$,

(b) *generic existence of nowhere dense ultrafilters.*

THEOREM D: *The following are equivalent:*
(a) $\mathrm{cof}(\mathcal{E},\mathcal{M}) = \mathfrak{c}$,
(b) *generic existence of measure zero ultrafilters.*

THEOREM E: *The following are equivalent:*
(a) $\mathfrak{d} = \mathfrak{c}$,
(b) *generic existence of countable closed ultrafilters,*
(c) *generic existence of ordinal ultrafilters,*
(d) *generic existence of P-points.*

Here, $\mathfrak{d}$ denotes as usual the **dominating number**, that is, the size of the smallest family of functions $\mathcal{F} \subseteq \omega^\omega$ such that for all $g \in \omega^\omega$ there is $f \in \mathcal{F}$ with $f(n) \geq g(n)$ for almost all n. In future, we shall abbreviate *for almost all n* by $\forall^\infty n$, and write $f \geq^* g$ for $\forall^\infty n(f(n) \geq g(n))$. Given downward closed families $\mathcal{I} \subseteq \mathcal{J} \subseteq P(X)$, define

$$\mathrm{cof}(\mathcal{I},\mathcal{J}) = \min\{|\mathcal{F}|; \mathcal{F} \subseteq \mathcal{J} \text{ and all } I \in \mathcal{I} \text{ are contained in some } J \in \mathcal{F}\}.$$

In case $\mathcal{I} = \mathcal{J}$, we write $\mathrm{cof}(\mathcal{I})$ for $\mathrm{cof}(\mathcal{I},\mathcal{I})$, and call it the **cofinality of** $\mathcal{I}$. In most applications, $\mathcal{I}$ and $\mathcal{J}$ are ideals. (Note that one always has $\mathrm{cof}(\mathcal{I},\mathcal{J}) \geq \mathrm{cof}(\langle\mathcal{I}\rangle,\langle\mathcal{J}\rangle) = \mathrm{cof}(\mathcal{I},\langle\mathcal{J}\rangle)$.) Also, if $\mathcal{I} \subseteq \mathcal{P}(X)$ is an ideal containing all singletons, let

$$\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{F}|; \mathcal{F} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{F} = X\},$$

the **covering number of** $\mathcal{I}$. (Clearly, $\mathrm{cov}(\mathcal{I}) = \mathrm{cof}(\mathcal{F},\mathcal{I})$ where $\mathcal{F}$ is the ideal of finite subsets of X.) Finally, $\mathcal{M}$ denotes the ideal of meager sets (the σ-ideal generated by $\mathcal{M}_0$) while $\mathcal{E}$ is the σ-ideal generated by $\mathcal{E}_0$.

Some of the proofs of the results discussed in the preceding paragraph involve new theorems about cardinal invariants of the continuum which are interesting in their own right. We single them out and — along with some basic results which we review — prove them in section 1.

THEOREM F: $\mathrm{cof}(\mathcal{E},\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{E})\}$.

THEOREM G: $\mathrm{cof}(\mathcal{C},\mathcal{M}_0) = \mathfrak{d} = \mathrm{cof}(\mathcal{J}_\alpha)$ *where* $\alpha \geq \omega^2$.

Here, given an ideal $\mathcal{I} \subseteq P(X)$ containing all singletons, let

$$\mathrm{non}(\mathcal{I}) = \min\{|F|; F \subseteq X \text{ and } F \notin \mathcal{I}\},$$

the **uniformity of** $\mathcal{I}$. $\mathrm{non}(\mathcal{I})$ is dual to $\mathrm{cov}(\mathcal{I})$ mentioned above. Also $\mathcal{J}_\alpha$ denotes the family of subsets of α of order type $< \alpha$.

We conclude our considerations in section 4 with a few additional results and some open problems.

We presuppose familiarity with forcing techniques. See [Je] for the basics and [BJ] whose notation we largely follow for more advanced material. We usually identify the rationals with $2^{<\omega}$ and construe $2^{<\omega}$ in this case as a subset of 2^ω in the obvious way: the set of infinite $0-1$-sequences which are eventually 0. If $s \in 2^{<\omega}$, let $[s] = \{f \in 2^\omega; s \subseteq f\}$ denote the clopen set determined by s; similarly if $T \subseteq 2^{<\omega}$ is a tree, let $[T] = \{f \in 2^\omega; f \restriction n \in T$ for all $n\}$ be the set of its **branches**. Further notation will be introduced as needed.

ACKNOWLEDGEMENT: I thank James Baumgartner for asking the questions which eventually led to this paper, and Albin Jones for an illuminating discussion which simplified and streamlined the approach to $\mathrm{cof}(\mathcal{J}_\alpha) = \mathfrak{d}$ taken in 1.6. I am also grateful to the referee for many suggestions which improved the presentation of this work.

1. Cofinalities of ideals

As mentioned in the introduction, we discuss several cofinalities of ideals which we shall need in the next section for the characterization of generic existence of various classes of ultrafilters.

1.1 Since some of the results of this section can be easily dualized to corresponding results about additivity numbers, we shall expound the latter as well even though we will not use them later on. Therefore some of the results in this section will be formulated in the language of **Galois–Tukey connections** (see [BJ, section 2.1] or [Ba1, section 2]). Let $\mathcal{I} \subseteq \mathcal{J}$ be ideals. We define

$$\mathrm{add}(\mathcal{I},\mathcal{J}) = \min\{|\mathcal{F}|; \mathcal{F} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{F} \notin \mathcal{J}\}.$$

In case $\mathcal{I} = \mathcal{J}$, we write $\mathrm{add}(\mathcal{I}) = \mathrm{add}(\mathcal{I},\mathcal{I})$, and call it the **additivity of** $\mathcal{I}$. $\mathrm{add}(\mathcal{I},\mathcal{J})$ is dual to $\mathrm{cof}(\mathcal{I},\mathcal{J})$ defined in the Introduction. Call a family of functions $\mathcal{F} \subseteq \omega^\omega$ **unbounded** if for all $g \in \omega^\omega$ there is $f \in \mathcal{F}$ which lies infinitely often above g. Then

$$\mathfrak{b} = \min\{|\mathcal{F}|; \mathcal{F} \subseteq \omega^\omega \text{ and } \mathcal{F} \text{ is unbounded}\}$$

is the **unbounding number** which is dual to $\mathfrak{d}$. Finally let $\mathcal{N}$ denote the σ-ideal of null subsets of 2^ω.

1.2. A CHARACTERIZATION OF CLOSED MEASURE ZERO SETS. Fix $h \in \omega^\omega$ strictly increasing. Put $k_0 = 0, \dots, k_{n+1} = k_n + h(n)$. Then $\mathcal{C}_h$ denotes the class of all functions S with domain ω and $S(n) \subseteq 2^{[k_n,k_{n+1})}$ for all n such that

$$\frac{|S(n)|}{2^{h(n)}} \leq \frac{1}{2^n}.$$

For $S \in \mathcal{C}_h$ we can form $F_S \subseteq 2^\omega$ as follows:

$$F_S = \{x \in 2^\omega; \forall^\infty n \ \ x \restriction [k_n, k_{n+1}) \in S(n)\}.$$

F_S is easily seen to be a countable union of the closed null sets

$$F_S^\ell = \{x \in 2^\omega; \forall n \geq \ell \ \ x \restriction [k_n, k_{n+1}) \in S(n)\}.$$

Thus $F_S \in \mathcal{E}$. A sort of converse is also true.

LEMMA 1 (Bartoszyński and Shelah [BSh], see also [BJ, Lemma 2.6.3]): *Given $C \in \mathcal{E}$, there are $h \in \omega^\omega$ and $S \in \mathcal{C}_h$ such that $C \subseteq F_S$.*

Of course, one also has (with the same argument)

LEMMA 2: *Given $C \in \mathcal{E}_0$, there are $h \in \omega^\omega$ and $S \in \mathcal{C}_h$ such that $C \subseteq F_S^0$.*

We shall use these crucial facts several times.

1.3 We next review a few well-known results concerning additivity and cofinality of $\mathcal{E}$ and $\mathcal{M}$. Note the analogy between our Theorem F and the following result.

THEOREM 1 (Miller and Truss, see [BJ, Corollary 2.2.9 and Theorem 2.2.11]): $\text{add}(\mathcal{M}) = \min\{\mathfrak{b}, \text{cov}(\mathcal{M})\}$ *and* $\text{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \text{non}(\mathcal{M})\}$.

The following is probably the deepest result in the area.

THEOREM 2 (Bartoszyński and Shelah [BSh], see also [BJ, Theorem 2.6.17]): $\text{add}(\mathcal{E}) = \text{add}(\mathcal{M})$ *and* $\text{cof}(\mathcal{E}) = \text{cof}(\mathcal{M})$.

When dealing with cofinalities, it does not matter whether we consider $\mathcal{M}$ or $\mathcal{M}_0$ ($\mathcal{E}$ or $\mathcal{E}_0$, respectively).

PROPOSITION:

(a) (Cichoń and Galvin, see [Mil, Theorem 4]) $\text{cof}(\mathcal{M}) = \text{cof}(\mathcal{M}_0, \mathcal{M}) = \text{cof}(\mathcal{M}_0)$,

(b) $\text{cof}(\mathcal{E}) = \text{cof}(\mathcal{E}_0, \mathcal{E}) = \text{cof}(\mathcal{E}_0)$,

(c) $\mathrm{cof}(\mathcal{E},\mathcal{M}) = \mathrm{cof}(\mathcal{E}_0,\mathcal{M}) = \mathrm{cof}(\mathcal{E}_0,\mathcal{M}_0)$.

Proof: (b) $\mathrm{cof}(\mathcal{E}) \geq \mathrm{cof}(\mathcal{E}_0,\mathcal{E})$ and $\mathrm{cof}(\mathcal{E}_0) \geq \mathrm{cof}(\mathcal{E}_0,\mathcal{E})$ are obvious. To see $\mathrm{cof}(\mathcal{E}) \leq \mathrm{cof}(\mathcal{E}_0,\mathcal{E})$, take an $\mathcal{E}_0$-cofinal $\mathcal{F} \subseteq \mathcal{E}$. By 1.2 we can assume the members of $\mathcal{F}$ are all of the form F_S for $S \in \mathcal{C}_h$ for some h. For an arbitrary $C \in \mathcal{E}$, find (1.2) $h', S' \in \mathcal{C}_{h'}$ with $C \subseteq F_{S'}$. Then find (by assumption on $\mathcal{F}$) $F_S \in \mathcal{F}$ with $F^0_{S'} \subseteq F_S$. Using the combinatorial definition of the F_S, we now see $F_{S'} \subseteq F_S$ and thus $C \subseteq F_S$.

To see $\mathrm{cof}(\mathcal{E}_0) \leq \mathrm{cof}(\mathcal{E}_0,\mathcal{E})$, we start similarly and argue instead that $F^0_{S'} \subseteq F_S$ must imply that $F^0_{S'} \subseteq F^\ell_S$ for some $\ell \in \omega$.

(c) Again $\mathrm{cof}(\mathcal{E},\mathcal{M}) \geq \mathrm{cof}(\mathcal{E}_0,\mathcal{M})$ and $\mathrm{cof}(\mathcal{E}_0,\mathcal{M}_0) \geq \mathrm{cof}(\mathcal{E}_0,\mathcal{M})$ are clear. The converse inequalities can be shown by taking over the arguments for (a) from [Mi1, pp. 113–114] almost verbatim. ∎

1.4. Proof of Theorem F. We shall deal simultaneously with both the additivity $\mathrm{add}(\mathcal{E},\mathcal{M})$ and the cofinality $\mathrm{cof}(\mathcal{E},\mathcal{M})$. First recall the following result.

Theorem 1 (Miller [Mi], [Mi1], see also [BJ, Theorem 2.6.10]): *There are functions $\omega^\omega \to \mathcal{E}$: $f \mapsto A_f$ and $\mathcal{M} \to \omega^\omega$: $B \mapsto g_B$ such that if $A_f \subseteq B$ then $f \leq^* g_B$.*

Corollary 1 (Miller): $\mathrm{add}(\mathcal{E},\mathcal{M}) \leq \mathfrak{b}$ *and* $\mathrm{cof}(\mathcal{E},\mathcal{M}) \geq \mathfrak{d}$.

Since the inequalities $\mathrm{add}(\mathcal{E},\mathcal{M}) \leq \mathrm{cov}(\mathcal{E})$ and $\mathrm{cof}(\mathcal{E},\mathcal{M}) \geq \mathrm{non}(\mathcal{E})$ are obvious, it suffices to prove the following result.

Theorem 2: *There are functions $\mathcal{E} \to \omega^\omega$: $A \mapsto f_A$, $\omega^\omega \times \mathcal{E} \to \mathcal{E}$: $(g, A) \mapsto B_{g,A}$, $\omega^\omega \times 2^\omega \times \mathcal{E}$: $(g, y, B) \mapsto h_{g,y,B}$ and $(\omega^\omega)^2 \times 2^\omega \to \mathcal{M}$: $(g, e, y) \mapsto C_{g,e,y}$ such that if $f_A \leq^* g$, $y \in 2^\omega \setminus B_{g,A}$ and $h_{g,y,B_{g,A}} \leq^* e$, then $A \subseteq C_{g,e,y}$.*

Corollary 2: $\mathrm{add}(\mathcal{E},\mathcal{M}) \geq \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{E})\}$ *and* $\mathrm{cof}(\mathcal{E},\mathcal{M}) \leq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{E})\}$.

Proof of Theorem 2: Let $A \in \mathcal{E}$. By 1.2, we can assume it is of the form F_S for some $S \in \mathcal{C}_{f_A}$ and $f_A \in \omega^\omega$, i.e.

$$A = F_S = \{x \in 2^\omega; \forall^\infty n \;\; x \restriction [k_n, k_{n+1}) \in S(n)\}$$

where $k_0 = 0, \ldots, k_{n+1} = k_n + f_A(n)$ and $S(n) \subseteq 2^{[k_n,k_{n+1})}$ satisfies

$$\frac{|S(n)|}{2^{f_A(n)}} \leq \frac{1}{2^n}.$$

Now note that if $g \geq^* f_A$, then we can easily find $T \in \mathcal{C}_{\bar{g}}$ such that $F_S \subseteq F_T$, where we recursively define $\bar{g}(0) = g(0)$ and $\bar{g}(n+1) = g(\bar{g}(n))$. Put $B =$

$B_{g,A} = F_T \in \mathcal{E}$ (in case $g \not\geq^* f_A$, the definition of $B_{g,A}$ is irrelevant). Let $\ell_0 = 0, \ldots, \ell_{n+1} = \ell_n + \bar{g}(n)$. If $y \notin B$, then there are infinitely many i with $y \restriction [\ell_i, \ell_{i+1}) \notin T(i)$. Thus we can find $h_{g,y,B}(n)$ such that for all n there is $n \leq i < h_{g,y,B}(n)$ such that $y \restriction [\ell_i, \ell_{i+1}) \notin T(i)$ (if $y \in B$, it does not matter how we define $h_{g,y,B} \in \omega^\omega$). Now, given $e \in \omega^\omega$, let $C_{g,e,y}$ be the set of all $x \in 2^\omega$ such that

$$x \restriction \bigcup_{n \leq i < e(n)} [\ell_i, \ell_{i+1}) \neq y \restriction \bigcup_{n \leq i < e(n)} [\ell_i, \ell_{i+1})$$

holds for almost all n. The meagerness of $C_{g,e,y}$ is obvious, and in case $e \geq^* h_{g,y,B_{g,A}}$, we see $A = F_S \subseteq F_T = B_{g,A} \subseteq C_{g,e,y}$ as follows: if $x \in F_T$, then $x \restriction [\ell_i, \ell_{i+1}) \in T(i)$ for almost all i. Since, for almost all n, we have $y \restriction [\ell_i, \ell_{i+1}) \notin T(i)$ for some $n \leq i < e(n)$, we have $x \restriction \bigcup_{n \leq i < e(n)} [\ell_i, \ell_{i+1}) \neq y \restriction \bigcup_{n \leq i < e(n)} [\ell_i, \ell_{i+1})$ for almost all n, as required. ∎

Proof of Corollary 2 from Theorem 2: Let $\mathcal{A} \subseteq \mathcal{E}$ be of size $< \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{E})\}$. Let g eventually dominate all f_A, $A \in \mathcal{A}$, let $y \in 2^\omega \setminus \bigcup\{B_{g,A}; A \in \mathcal{A}\}$, and let e eventually dominate all $h_{g,y,B_{g,A}}$, $A \in \mathcal{A}$. Then clearly $\bigcup \mathcal{A} \subseteq C_{g,e,y} \in \mathcal{M}$, as required. This shows the first inequality.

To prove the second, let $\mathcal{F}$ be a dominating family of size $\mathfrak{d}$, and let $Y \notin \mathcal{E}$ of size $\mathrm{non}(\mathcal{E})$. Given $A \in \mathcal{E}$, first find $g \in \mathcal{F}$ with $f_A \leq^* g$, then $y \in Y \setminus B_{g,A}$, then $e \in \mathcal{F}$ with $h_{g,y,B_{g,A}} \leq^* e$, which means that $A \subseteq C_{g,e,y}$. Thus $\{C_{g,e,y}; g, e \in \mathcal{F}, y \in Y\}$ is cofinal for $\mathcal{E}$. ∎

We note at this point that while $\mathfrak{d} \leq \mathrm{cof}(\mathcal{E}, \mathcal{M}) \leq \mathrm{cof}(\mathcal{M})$ and $\mathrm{add}(\mathcal{M}) \leq \mathrm{add}(\mathcal{E}, \mathcal{M}) \leq \mathfrak{b}$, none of these cardinals are provably equal. For the cofinalities, this will follow from stronger results exhibited in section 3, and for the additivity numbers, we can argue as follows. If V is gotten by a countable support iteration of ω_2 Laver reals over a model of CH, it satisfies $\omega_2 = \mathfrak{b} > \mathrm{add}(\mathcal{E}, \mathcal{M}) = \mathrm{cov}(\mathcal{E}) = \omega_1$. This is so because dominating reals are added and each new real belongs to a closed measure zero set of the ground model. On the other hand, if we add both Laver and random reals in the iteration, we get $\omega_2 = \mathrm{add}(\mathcal{E}, \mathcal{M}) > \mathrm{add}(\mathcal{M}) = \omega_1$ because the random reals yield $\mathrm{cov}(\mathcal{N}) = \omega_2$, a fortiori $\mathrm{cov}(\mathcal{E}) = \omega_2$, and $\mathrm{add}(\mathcal{E}, \mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{E})\} = \omega_2$, while no Cohen reals are adjoined. See [BJ, chapter 7] for more on this.

1.5. PROOF THAT $\mathrm{cof}(\mathcal{C}, \mathcal{M}_0) = \mathfrak{d}$. We proceed to the first half of the proof of Theorem G. One inequality is easy.

PROPOSITION: $\mathrm{cof}(\mathcal{C}, \mathcal{M}_0) \geq \mathfrak{d}$.

Proof: Fix $y \in 2^\omega$. Given $f \in \omega^\omega$, we produce A_f with unique limit point y by selecting $y_s \in [s]$ for each $s \in 2^{<\omega}$ such that, for some $k \in \omega$, we have $k = \min\{i; s(i) \neq y(i)\}$ and $|s| = f(k)$; let A_f be the set of those y_s together with y. On the other hand, given $B \subseteq 2^\omega$ nowhere dense, define $g_B \in \omega^\omega$ by

$$g_B(k) = \min\{|s|; [s] \cap B = \emptyset \text{ and } k = \min\{i; s(i) \neq y(i)\}\}.$$

We verify easily that $A_f \subseteq B$ implies $g_B \geq f$.

Thus, if $\mathcal{B} \subseteq \mathcal{M}_0$ is cofinal for $\mathcal{C}$, then $\{g_B; B \in \mathcal{B}\}$ is dominating, and we're done. ∎

The other direction is harder and involves a Cantor–Bendixson rank argument.

THEOREM: *There are functions* $\omega^\omega \times \omega_1 \to \mathcal{M}_0$: $(f, \alpha) \mapsto B_{f,\alpha}$, $\mathcal{C} \to \omega_1$: $C \mapsto \beta_C$, $\mathcal{C} \to \omega^\omega$: $C \mapsto g_C$ *and* $\mathcal{C} \times \omega_1 \to \omega^\omega$: $(C, \alpha) \mapsto h_{C,\alpha}$ *such that if* $\beta_C < \alpha$ *and* $g_C, h_{C,\alpha} \leq f$, *then* $C \subseteq B_{f,\alpha}$. *In particular,* $\mathrm{cof}(\mathcal{C}, \mathcal{M}_0) \leq \mathfrak{d}$.

Proof: Given a limit ordinal $\lambda < \omega_1$, fix once and for all a strictly increasing sequence $\langle \eta_n^\lambda; n \in \omega \rangle$ converging to λ. Given $s \in 2^{<\omega}$, let $\bar{s}$ denote the set of its proper initial segments. Put $\Phi := \{\phi: \bar{s} \to \omega; s \in 2^{<\omega}\}$. Also, given $s \in 2^{<\omega}$ and $k > |s|$, let $s^k \supseteq s$ be such that $|s^k| = k$, $s^k(i) = 0$ for $|s| \leq i < k-1$ and $s^k(k-1) = 1$. Finally, if C is a closed set, T_C is the tree associated with C (i.e. $C = [T_C]$).

Let f: $2^{<\omega} \to \omega$ and $0 < \alpha < \omega_1$. To avoid pathologies assume

(I) $f(s) > |s|$,

(II) if $s \subset t$ then $f(s) < f(t)$.

We can construe f also as a function $\bar{f}$: $\Phi \to \omega$ (by fixing once and for all a bijection $2^{<\omega} \to \Phi$). Increasing f further if necessary, we may suppose as well that

(III) $f(s) \geq \bar{f}(f \restriction \bar{s})$ for all s.

We define a rank function $\rho = \rho_{f,\alpha}$: $2^{<\omega} \to \omega_1$ such that

(i) $\rho(\langle\rangle) = \alpha$,

(ii) $s \subseteq t$ implies $\rho(t) \leq \rho(s)$,

(iii) if $\rho(s) = \gamma + 1$, then $\rho(s^{f(s)}) = \gamma$,

(iv) if $\rho(s) = \lambda$ is a limit, then $\rho(s^{f(s)}) = \eta^\lambda_{f(s)}$,

(v) each node has the maximal possible rank subject to (i) through (iv).

Let $S = S_{f,\alpha}$ be the tree consisting of all nodes of rank > 0, and put $B = B_{f,\alpha} = [S]$. Notice that B is nowhere dense by (iii) and (iv).

Given $C \subseteq 2^\omega$ countable and closed, let $\tilde{\beta}$ be its Cantor–Bendixson rank, i.e. the least ordinal such that the $\tilde{\beta}$-th derivative $C^{\tilde{\beta}}$ is empty. By compactness,

$\tilde{\beta} = \beta + 1$ is a successor ordinal and C^β is finite. Put $\beta_C = \beta$, and let ζ_C be the following rank function on $2^{<\omega}$:

$$\zeta_C(s) = \begin{cases} \sup\{\gamma; C^\gamma \cap [s] \neq \emptyset\} = \max\{\gamma; C^\gamma \cap [s] \neq \emptyset\} & \text{for } s \in T_C, \\ -1 & \text{for } s \in 2^{<\omega} \setminus T_C. \end{cases}$$

The equality of max and sup holds by compactness. Now define g_C: $2^{<\omega} \to \omega$ such that

(A) $\zeta_C(s^k) < \zeta_C(s)$ for all $k \geq g_C(s)$ and all $s \in T_C$.

Note that if $s \in 2^{<\omega}$, the information in ϕ: $\bar{s} \to \omega$ suffices to compute $\rho_{\phi,\alpha}(s)$ according to (i) through (v). Hence, given $\alpha > \beta_C$, we can define $h = h_{C,\alpha}$: $\Phi \to \omega$ such that

(B) if ϕ: $\bar{s} \to \omega$ and $\rho_{\phi,\alpha}(s) = \lambda$ is limit with $\lambda > \zeta_C(s)$, then $\eta^\lambda_{h_{C,\alpha}(\phi)} > \zeta_C(s)$.

(For other ϕ's, the value of $h_{C,\alpha}(\phi)$ does not matter.)

Now assume that $C \subseteq 2^\omega$ is countable closed and $\alpha > \beta_C$, $f \geq g_C$ and $\bar{f} \geq h_{C,\alpha}$. We argue that $\zeta_C(s) < \rho_{f,\alpha}(s)$ for all $s \in 2^{<\omega}$. Note that this immediately implies $T_C \subseteq S_{f,\alpha}$, and thus $C \subseteq B_{f,\alpha}$ so that the first half of the theorem is proved.

This goes by induction: the basic step for $\langle\rangle$ is immediate by $\alpha > \beta_C$. So let $s \neq \langle\rangle$ and assume $\zeta_C(t) < \rho_{f,\alpha}(t)$ holds for all $t \subset s$. If s is not of the form $t^{f(t)}$ for $t \subset s$, we have $\rho_{f,\alpha}(s) = \rho_{f,\alpha}(s \restriction (|s| - 1))$ and the conclusion is obvious. If $s = t^{f(t)}$, $t \subset s$, then: either $\rho_{f,\alpha}(t) = \gamma + 1$ and we have $\rho_{f,\alpha}(s) = \gamma > \zeta_C(s)$ by induction hypothesis, (A) and $f(t) \geq g_C(t)$; or $\rho_{\phi,\alpha}(t) = \rho_{f,\alpha}(t) = \lambda$ is a limit (with $\phi = f \restriction \bar{t}$), and we have $\rho_{f,\alpha}(s) = \eta^\lambda_{f(t)} \geq \eta^\lambda_{\bar{f}(\phi)} \geq \eta^\lambda_{h_{C,\alpha}(\phi)} > \zeta_C(t) \geq \zeta_C(s)$ by induction hypothesis, (B), (III) and $\bar{f}(\phi) \geq h_{C,\alpha}(\phi)$. Hence the argument is complete.

To see the second half of the theorem, take a dominating family $\mathcal{F}$ of size $\mathfrak{d}$, put $\mathcal{A} = \{B_{f,\alpha}; f \in \mathcal{F} \text{ and } \alpha < \omega_1\}$ and note that every $C \in \mathcal{C}$ is contained in a member of $\mathcal{A}$, by the first part. ∎

It is important to phrase our result with $\mathcal{M}_0$ because it is well-known that $\text{add}(\mathcal{C}, \mathcal{M}) = \text{non}(\mathcal{M})$ and that $\text{cof}(\mathcal{C}, \mathcal{M}) = \text{cov}(\mathcal{M})$. (The former is obvious, and the easiest way to see the latter is to use Bartoszyński's characterization of $\text{cov}(\mathcal{M})$, see [BJ, Theorem 2.4.1].)

1.6. Proof that $\text{cof}(\mathcal{J}_\alpha) = \mathfrak{d}$. We prove something more general. For ordinals $\alpha \geq \beta$ let $\mathcal{J}^\alpha_\beta$ denote the family of subsets of α of order type $< \beta$. Then $\mathcal{J}^\alpha_\beta$ is an ideal iff β is indecomposable, and $\mathcal{J}^\alpha_\alpha = \mathcal{J}_\alpha$. Now let $\alpha \geq \omega^2$ be indecomposable and countable. By $\mathcal{B}_\alpha$ we denote the bounded subsets of α. For $\beta \leq \gamma$, define

$$\mathrm{add}^\star(\mathcal{J}^\alpha_\beta,\mathcal{J}^\alpha_\gamma) =$$
$$\min\{|\mathcal{F}|; \mathcal{F}\subseteq \mathcal{J}^\alpha_\beta \text{ and no member of } \mathcal{J}^\alpha_\gamma \text{ contains all sets of } \mathcal{F} \text{ modulo } \mathcal{B}_\alpha\}.$$

Put $\mathrm{add}^\star(\mathcal{J}^\alpha_\beta) = \mathrm{add}^\star(\mathcal{J}^\alpha_\beta,\mathcal{J}^\alpha_\beta)$. Let us notice that $\mathrm{add}^\star$ can be considered a dual version of cof, for if we define the obvious dual version of $\mathrm{add}^\star$,

$$\mathrm{cof}^\star(\mathcal{J}^\alpha_\beta,\mathcal{J}^\alpha_\gamma) =$$
$$\min\{|\mathcal{F}|; \mathcal{F}\subseteq \mathcal{J}^\alpha_\gamma \text{ and every member of } \mathcal{J}^\alpha_\beta \text{ lies in a set of } \mathcal{F} \text{ modulo } \mathcal{B}_\alpha\},$$

then one has $\mathrm{cof}^\star(\mathcal{J}^\alpha_\beta,\mathcal{J}^\alpha_\gamma) = \mathrm{cof}(\mathcal{J}^\alpha_\beta,\mathcal{J}^\alpha_\gamma)$ for $\omega+1\le\beta\le\gamma\le\alpha<\omega_1$ where α is as usual indecomposable (this is immediate for $\alpha=\beta=\gamma$; for the general case, see Corollary 3 below). Of course, this may fail if α is not indecomposable.

We first show

PROPOSITION 1: *Let $\alpha\ge\omega^2$ be indecomposable and countable. Then there are maps $\omega^\omega\to\mathcal{J}^\alpha_{\omega+1}$: $f\mapsto A_f$ and $\mathcal{J}_\alpha\to\omega^\omega$: $B\mapsto g_B$ such that if $A_f\subseteq B$ modulo $\mathcal{B}_\alpha$ then $g_B\ge^* f$.*

COROLLARY 1: *$\mathrm{cof}^\star(\mathcal{J}^\alpha_{\omega+1},\mathcal{J}_\alpha)\ge\mathfrak{d}$ and $\mathrm{add}^\star(\mathcal{J}^\alpha_{\omega+1},\mathcal{J}_\alpha)\le\mathfrak{b}$ for countable and indecomposable $\alpha\ge\omega^2$.*

Proof of Proposition 1: By assumption, α can be written as a disjoint increasing union of countable intervals I_n, $n\in\omega$, of ordinals (i.e. $\sup(I_n)=\min(I_{n+1})$ for all n) such that o.t.$I_{n+1}\ge$ o.t.I_n for all n. Fix bijections $h_n\colon\omega\to I_n$.

Given $f\in\omega^\omega$, let $A_f=\{h_n(i); n\in\omega \text{ and } i<f(n)\}$. Clearly o.t.$A_f=\omega$. On the other hand, given $B\subseteq\alpha$ with o.t.$B<\alpha$, we know that o.t.$(B\cap I_n)<$ o.t.I_n for almost all n. For each such n, choose $g_B(n)$ such that $h_n(g_B(n))\notin B$ (for other n, $g_B(n)$ is irrelevant). Now assume that $A_f\subseteq B$ modulo $\mathcal{B}_\alpha$. Let n be such that o.t.$(B\cap I_n)<$ o.t.I_n and $A_f\cap I_n\subseteq B\cap I_n$. If we had $g_B(n)<f(n)$, then $h_n(g_B(n))\in A_f\setminus B$, a contradiction. Hence $g_B(n)\ge f(n)$, as required. ∎

Proof of Corollary 1 from Proposition 1: Given $\mathcal{B}\subseteq\mathcal{J}_\alpha$ cofinal for $\mathcal{J}^\alpha_{\omega+1}$ (modulo $\mathcal{B}_\alpha$), let $\mathcal{G}=\{g_B; B\in\mathcal{B}\}$ and note that $\mathcal{G}$ must be dominating. Hence $|\mathcal{B}|\ge\mathfrak{d}$.

Given $\mathcal{F}\subseteq\omega^\omega$ unbounded, let $\mathcal{A}=\{A_f; f\in\mathcal{F}\}$. Choose $B\in\mathcal{J}_\alpha$. By the Proposition, there must be $f\in\mathcal{F}$ such that A_f is not contained in B modulo $\mathcal{B}_\alpha$. Hence $\mathrm{add}^\star(\mathcal{J}^\alpha_{\omega+1},\mathcal{J}_\alpha)\le|\mathcal{F}|$. ∎

THEOREM: *Let $\alpha \geq \beta \geq \omega^2$ be both indecomposable and countable. There are maps $\omega^\omega \to \mathcal{J}^\alpha_\beta$: $f \mapsto B^f$ and $\mathcal{J}^\alpha_\beta \to \omega^\omega$: $C \mapsto g_C$ such that if $f \geq g_C$, then $C \subseteq B^f$. If in addition $\beta = \omega^\delta$ for δ non-limit, we also have that if $f \geq^* g_C$, then $C \subseteq B^f$ modulo $\mathcal{B}_\alpha$.*

COROLLARY 2: *For $\alpha \geq \beta \geq \omega^2$ both countable and indecomposable, we have $\mathrm{cof}(\mathcal{J}^\alpha_\beta) \leq \mathfrak{d}$. If $\beta = \omega^\delta$, δ non-limit, we also have $\mathrm{add}^\star(\mathcal{J}^\alpha_\beta) \geq \mathfrak{b}$.*

Note that if $\beta = \omega^\lambda$ for λ limit, then $\mathrm{add}^\star(\mathcal{J}^\alpha_\beta) = \omega$ is easy to see.

Proof of Theorem: The argument is quite similar to the one used for Theorem 1.5. Given a limit ordinal $\lambda < \omega_1$ fix once and for all a strictly increasing sequence $\langle \eta^\lambda_n \geq 2; n \in \omega \rangle$ converging to λ. Let $\beta = \omega^\delta \leq \alpha = \omega^\gamma$, $\gamma \geq \delta \geq 2$, be indecomposable. We produce recursively a well-founded tree $T \subseteq \omega^{<\omega}$ and sets $\langle A_\sigma; \sigma \in T \rangle$ such that

(i) $A_{\langle\rangle} = \omega^\gamma$,
(ii) all A_σ are intervals of ordinals and have o.t. ω^θ for some $1 \leq \theta \leq \gamma$,
(iii) if σ is a terminal node in T, then o.t.$A_\sigma = \omega$,
(iv) if σ is not a terminal node in T, then A_σ is the disjoint increasing union of the $A_{\sigma\hat{}i}$,
(v) if o.t.$A_\sigma = \omega^{\theta+1}$, then o.t.$A_{\sigma\hat{}i} = \omega^\theta$ for all i,
(vi) if o.t.$A_\sigma = \omega^\lambda$ for λ limit, then o.t.$A_{\sigma\hat{}i} = \omega^{\eta^\lambda_i}$.

It is clear that this can be done. Also, for $\sigma \in T$, let $\bar\sigma \subseteq T$ be the collection of strict initial segments of σ; put $\Phi := \{\phi: \bar\sigma \to \omega; \sigma \in T\}$. Finally, given $f: T \to \omega$, define recursively $\zeta^f = \zeta: T \to \omega_1$ such that

(a) $\zeta(\langle\rangle) = \delta$,
(b) if $\zeta(\sigma) = \theta + 1$ and $i < f(\sigma)$, then $\zeta(\sigma\hat{}i) = \theta + 1$,
(c) if $\zeta(\sigma) = \theta + 1$ and $i \geq f(\sigma)$, then $\zeta(\sigma\hat{}i) = \theta$,
(d) if $\zeta(\sigma) = \lambda$ for λ limit, then $\zeta(\sigma\hat{}i) = \eta^\lambda_{f(\sigma)}$ for all i.

Now, fix $f: T \to \omega$. With f we plan to associate a set $B = B^f \subseteq \omega^\gamma$ such that o.t.$B^f < \omega^\delta$. Note that we can construe f also as a map $\bar f: \Phi \to \omega$. By increasing f if necessary, we may assume that

(A) $f(\sigma) \geq \bar f(f \restriction \bar\sigma)$ for all $\sigma \in T$.

Next find (by backwards recursion) sets $B_\sigma \subseteq A_\sigma$, $\sigma \in T$, such that o.t.$B_\sigma < \omega^{\zeta(\sigma)}$ (where $\zeta = \zeta^f$), as follows:

(B) if σ is a terminal node, then

$$B_\sigma = \begin{cases} \emptyset & \text{if } \zeta(\sigma) = 0, \\ \text{the first } f(\sigma) \text{ elements of } A_\sigma & \text{if } \zeta(\sigma) = 1, \\ A_\sigma & \text{if } \zeta(\sigma) > 1; \end{cases}$$

(C) if σ is not a terminal node, then $B_\sigma = \bigcup_i B_{\sigma\hat{}i}$.

Using (b) through (d) above, we see inductively that o.t.$B_\sigma < \omega^{\zeta(\sigma)}$ is indeed true. Hence the set $B^f = B_{\langle\rangle}$ is as required.

Let $C \subseteq \omega^\gamma$ with o.t.$C < \omega^\delta$ be given. For $\sigma \in T$ put $C_\sigma = A_\sigma \cap C$. We shall define a function $g_C = g\colon T \to \omega$ by recursion such that

(I) o.t.$C_\sigma < \omega^{\zeta^g(\sigma)}$ for all $\sigma \in T$,

(II) if σ is a terminal node and $\zeta^g(\sigma) = 1$, then C_σ is contained in the first $g(\sigma)$ elements of A_σ.

This is done easily: the fact that o.t.$C < \omega^\delta$ and clause (a) take care of the case $\sigma = \langle\rangle$; clauses (b) through (d) then tell us how to produce g on the nodes which are not terminal. (II) says what to do on terminal nodes. We also construct an auxiliary function $\bar{g}_C = \bar{g}\colon \Phi \to \omega$: given $\phi \in \Phi$ with domain $\bar{\sigma}$ we can define $\zeta^\phi(\sigma)$ as in (a) through (d) above because this recursive definition involves only strict initial segments of σ. Now, define $\bar{g}\colon \Phi \to \omega$ such that

(III) if $\zeta^\phi(\sigma) = \lambda$ is a limit and $\zeta^g(\sigma) < \lambda$, then $\eta^\lambda_{\bar{g}(\phi)} \geq \zeta^g(\sigma)$

(in all other cases, the definition of $\bar{g}$ is irrelevant).

This completes our constructions and we are left with checking that whenever $C \subseteq \omega^\gamma$ has o.t. $< \omega^\delta$ and $f \geq g_C$, $\bar{f} \geq \bar{g}_C$, then $C \subseteq B^f$. (Note that the assumption $\bar{f} \geq \bar{g}_C$ is not in the statement of the theorem because we may think of f and $\bar{f}$ (g_C and $\bar{g}_C$, resp.) as being just one function.) Let us argue first by induction that one has $\zeta^f \geq \zeta^{g_C}$ everywhere. Indeed, for $\sigma = \langle\rangle$ this is clear by (a); if $\zeta^f(\sigma)$ is a successor, it follows easily for $\sigma\hat{}i$ from clauses (b) through (d); finally, if $\zeta^f(\sigma) = \lambda$ is a limit, it is clear for $\sigma\hat{}i$ in case $\zeta^g(\sigma) = \lambda$ from (d), and in case $\zeta^g(\sigma) < \lambda$, it is immediate from (d), (A) and (III). Now the proof that $C \subseteq B^f = B$ can be easily accomplished by backwards induction, showing that $C_\sigma \subseteq B_\sigma$. Using that $f \geq g_C$ everywhere and that $\zeta^f \geq \zeta^{g_C}$ everywhere, this induction is trivial by clauses (II), (B) and (C).

Similarly, one shows that if $f \geq^* g_C$, $\bar{f} \geq^* \bar{g}_C$ and δ is not a limit, then $C \subseteq B^f$ modulo $\mathcal{B}_\alpha$ by first proving that there is i such that $\zeta^f(\sigma) \geq \zeta^{g_C}(\sigma)$ for all σ with $\langle j\rangle \subseteq \sigma$ where $j \geq i$ — and hence $\bigcup_{j\geq i} C_{\langle j\rangle} \subseteq \bigcup_{j\geq i} B^f_{\langle j\rangle}$. This completes the proof of the theorem. ∎

Proof of Corollary 2 from Theorem: Given $\mathcal{F} \subseteq \omega^\omega$ dominating (everywhere), let $\mathcal{B} = \{B^f; f \in \mathcal{F}\}$, and note that $\mathcal{B}$ must be cofinal. Hence $|\mathcal{F}| \geq \mathrm{cof}(\mathcal{J}^\alpha_\beta)$.

Given $\mathcal{C} \subseteq \mathcal{J}^\alpha_\beta$ not $\star$-additive, let $\mathcal{G} = \{g_C; C \in \mathcal{C}\}$. Now if $f \geq^* g_C$ for all $C \in \mathcal{C}$, then $C \subseteq B^f$ modulo $\mathcal{B}_\alpha$ for all C, a contradiction. Hence $\mathcal{G}$ is unbounded, and $|\mathcal{C}| \geq \mathfrak{b}$. ∎

Summing up Corollaries 1 and 2 and generalizing them a little further, we get

COROLLARY 3: *If $\omega+1 \leq \beta \leq \gamma \leq \alpha$ and α is indecomposable and countable, then* $\mathrm{cof}(\mathcal{J}^\alpha_\beta, \mathcal{J}^\alpha_\gamma) = \mathrm{cof}^\star(\mathcal{J}^\alpha_\beta, \mathcal{J}^\alpha_\gamma) = \mathfrak{d}$. *Unless $\beta = \gamma = \omega^\lambda$ for λ limit, we additionally have* $\mathrm{add}^\star(\mathcal{J}^\alpha_\beta, \mathcal{J}^\alpha_\gamma) = \mathfrak{b}$.

Proof: For an ordinal β, let $S(\beta)$ be the least indecomposable ordinal $\geq \beta$. Then, by Corollaries 1 and 2, we have

$$\mathfrak{b} \geq \mathrm{add}^\star(\mathcal{J}^\alpha_{\omega+1}, \mathcal{J}_\alpha) \geq \mathrm{add}^\star(\mathcal{J}^\alpha_\beta, \mathcal{J}^\alpha_\gamma) \geq \mathrm{add}^\star(\mathcal{J}^\alpha_\beta) \geq \mathrm{add}^\star(\mathcal{J}^\alpha_{\mathrm{S}(\beta)}) \geq \mathfrak{b}$$

in case $\beta \neq \omega^\lambda$ for λ limit. Here, the inequality $\mathrm{add}^\star(\mathcal{J}^\alpha_\beta) \geq \mathrm{add}^\star(\mathcal{J}^\alpha_{\mathrm{S}(\beta)})$ holds since we work modulo $\mathcal{B}_\alpha$. Similarly, one argues that $\mathrm{add}^\star(\mathcal{J}^\alpha_\beta, \mathcal{J}^\alpha_\gamma) \geq \mathrm{add}^\star(\mathcal{J}^\alpha_{\mathrm{S}(\beta+1)})$ in case $\beta < \gamma$.

Dealing with the cofinalities is a little harder because of the non-starred version. First show $\mathrm{cof}(\mathcal{J}^\alpha_\beta) \leq \mathfrak{d}$ for $\omega+1 \leq \beta \leq \alpha$ and α indecomposable. Let $\beta = \beta_0 + \cdots + \beta_n$, $\beta_0 \geq \cdots \geq \beta_n$, be the decomposition of β into indecomposable ordinals. We proceed by induction on n. The case $n=0$ is dealt with in Corollary 2. So assume $n>0$. We clearly have $\mathrm{cof}(\mathcal{J}^\alpha_\beta) \leq \max\{\mathrm{cof}(\mathcal{J}^{\mathrm{S}(\beta)}_\beta), \mathrm{cof}(\mathcal{J}^\alpha_{\mathrm{S}(\beta)})\}$. Thus, using again Corollary 2, it suffices to show $\mathrm{cof}(\mathcal{J}^{\mathrm{S}(\beta)}_\beta) \leq \mathfrak{d}$. Note that $S(\beta) = \beta_0 \cdot \omega$. By Corollary 2 and induction, find $\mathcal{F}_0 \subseteq \mathcal{J}_{\mathrm{S}(\beta)}$, $\mathcal{F}_1 \subseteq \mathcal{J}_{\beta_0}$ and $\mathcal{F}_2 \subseteq \mathcal{J}^{S(\beta)}_{\beta_1+\cdots+\beta_n}$ cofinal of size $\leq \mathfrak{d}$. Let $\mathcal{F}$ consist of all sets $F \subseteq S(\beta)$ such that *either* for some $i \geq 0$, $F \cap [\beta_0 \cdot j, \beta_0 \cdot (j+1))$ belongs to (the isomorphic copy of) $\mathcal{F}_1$ for all $j < i$, $F \cap [\beta_0 \cdot i, \beta_0 \cdot (i+1)) = [\beta_0 \cdot i, \beta_0 \cdot (i+1))$ and $F \cap [\beta_0 \cdot (i+1), S(\beta)) = G \cap [\beta_0 \cdot (i+1), S(\beta))$ for some $G \in \mathcal{F}_2$, *or* for some $i \geq 0$, $F \cap [\beta_0 \cdot j, \beta_0 \cdot (j+1))$ belongs to $\mathcal{F}_1$ for all $j < i$ and $F \cap [\beta_0 \cdot i, S(\beta)) = G \cap [\beta_0 \cdot i, S(\beta))$ for some $G \in \mathcal{F}_0$ such that o.t.$G \cap [\beta_0 \cdot i, S(\beta)) = \beta_0$. To see $\mathcal{F} \subseteq \mathcal{J}^{S(\beta)}_\beta$ is cofinal, take $A \in \mathcal{J}^{S(\beta)}_\beta$. Increasing A, if necessary, we may assume it *either* has order type $> \beta_0$ in which case we can find i such that o.t.$A \cap \beta_0 \cdot (i+1) = \beta_0$ and thus A is included in a set of the first kind above, *or* it has order type β_0 and is cofinal in $S(\beta)$ and thus is included in a set of the second kind above. Hence we're done with the proof of $\mathrm{cof}(\mathcal{J}^\alpha_\beta) \leq \mathfrak{d}$.

By this, as well as Corollary 1, we now see

$$\mathfrak{d} \leq \mathrm{cof}^\star(\mathcal{J}^\alpha_{\omega+1}, \mathcal{J}_\alpha) \leq \mathrm{cof}^\star(\mathcal{J}^\alpha_\beta, \mathcal{J}^\alpha_\gamma) \leq \mathrm{cof}(\mathcal{J}^\alpha_\beta, \mathcal{J}^\alpha_\gamma) \leq \mathrm{cof}(\mathcal{J}^\alpha_\beta) \leq \mathfrak{d}. \qquad \blacksquare$$

Let us finally drop the indecomposability requirement on α. Then the starred cardinals may not make sense anymore — however, we can still investigate the cofinality.

PROPOSITION 2: *Let $\omega+1 \le \beta \le \gamma \le \alpha \le \omega_1$ where β and γ are countable. Assume $\alpha = \alpha_0 + \cdots + \alpha_{n-1}$, $\alpha_0 \ge \cdots \ge \alpha_{n-1}$, is the decomposition of α into indecomposable ordinals. Suppose that $\alpha_0+\cdots+\alpha_{k-1} < \gamma = \alpha_0+\cdots+\alpha_{k-1}+\gamma' \le \alpha_0+\cdots+\alpha_k$ for some $0 \le k \le n-1$. Then*

$$\operatorname{cof}(\mathcal{J}^\alpha_\beta, \mathcal{J}^\alpha_\gamma) = \begin{cases} \omega & \text{if } \alpha < \omega^2 \text{ or } \beta \le \min\{\omega k + \gamma', \omega(k+1)\}, \\ \mathfrak{d} & \text{otherwise.} \end{cases}$$

Proof: Put $\alpha'_i = \alpha_0 + \cdots + \alpha_i$ for $i < n$. So $\alpha = \alpha'_{n-1}$. Also let $\alpha'_{-1} = 0$.

The case $\alpha < \omega^2$ is easy. So let us assume $\beta \le \min\{\omega k + \gamma', \omega(k+1)\}$. Let $\mathcal{F}$ consist of all subsets of γ which contain k of the n intervals $[\alpha'_{i-1}, \alpha'_i)$, $i < n$, and less than $\min\{\gamma', \omega\}$ many points of each of the remaining intervals. Since the α_i are decreasing, each set in $\mathcal{F}$ has order type $< \gamma$, and thus $\mathcal{F} \subseteq \mathcal{J}^\alpha_\gamma$ is a countable subfamily. Furthermore, each set of order type $< \beta \le \min\{\omega k + \gamma', \omega(k+1)\}$ is contained in a member of $\mathcal{F}$. Hence $\operatorname{cof}(\mathcal{J}^\alpha_\beta, \mathcal{J}^\alpha_\gamma) \le \omega$ follows. The converse inequality is obvious.

Now suppose that $\alpha \ge \omega^2$ as well as $\beta > \min\{\omega k + \gamma', \omega(k+1)\}$. This means in particular $\alpha'_0 = \alpha_0 \ge \omega^2$. Fix $\mathcal{F} \subseteq \mathcal{J}^\alpha_\gamma$ cofinal for $\mathcal{J}^\alpha_\beta$. For each $i \le k$ with $\alpha_i \ge \omega^2$ define $\mathcal{F}_i$ to be the set of all $F_i = F \cap [\alpha'_{i-1}, \alpha'_i)$ where $F \in \mathcal{F}$ and o.t.$F_i < \alpha_i$. Clearly, each such F_i can be thought of as a subset of α_i. Then $\mathcal{F}_i \subseteq \mathcal{J}_{\alpha_i}$.

We claim one of the $\mathcal{F}_i$ is cofinal for $\mathcal{J}^{\alpha_i}_{\omega+1}$. For assume this were not the case. Then for each $i < k$ with $\alpha_i \ge \omega^2$ select $B_i \subseteq [\alpha'_{i-1}, \alpha'_i)$ of o.t.ω which is not contained in any of the F_i. For $i < k$ with $\alpha_i \le \omega$ let $B_i = [\alpha'_{i-1}, \alpha'_i)$. In case $\gamma' \ge \omega$, select B_k in a similar fashion. If $\gamma' < \omega$, let $B_k = [\alpha'_{k-1}, \gamma)$. Put $B = \bigcup_{i \le k} B_i$. Then o.t.$B \le \min\{\omega k + \gamma', \omega(k+1)\}$, and so $B \in \mathcal{J}^\alpha_\beta$. Hence there is $F \in \mathcal{F}$ with $B \subseteq F$. Since o.t.$F < \gamma$, there must be i with o.t.$F_i < \alpha_i$ where $F_i = F \cap [\alpha'_{i-1}, \alpha'_i)$. As $B \subseteq F$, we see $\alpha_i \ge \omega^2$, a contradiction. Hence the claim is proved.

Since one of the $\mathcal{F}_i$ is cofinal for $\mathcal{J}^{\alpha_i}_{\omega+1}$, we see

$$\operatorname{cof}(\mathcal{J}^\alpha_\beta, \mathcal{J}^\alpha_\gamma) \ge \operatorname{cof}(\mathcal{J}^{\alpha_i}_{\omega+1}, \mathcal{J}_{\alpha_i}) \ge \mathfrak{d}$$

by Corollary 1. On the other hand, we easily see

$$\operatorname{cof}(\mathcal{J}^\alpha_\beta, \mathcal{J}^\alpha_\gamma) \le \sup_{\alpha_i, \delta}\{\operatorname{cof}(\mathcal{J}^{\alpha_i}_\delta)\} \le \mathfrak{d}$$

by Corollary 3. ∎

2. Generic existence

Using the results of the previous section, we can prove Theorems C through E rather easily.

2.1. Proof of Theorem C. The implication (b) $\Longrightarrow$ (a) is straightforward: if $\mathrm{cof}(\mathcal{M}) < \mathfrak{c}$, then we can take a filter $\mathcal{F}$ on $2^{<\omega}$ (i.e. on the rationals) of size $< \mathfrak{c}$ such that $\mathcal{F} \supseteq \{A; 2^{<\omega} \setminus A \in \mathcal{M}_0\}$ because $\mathrm{cof}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}_0)$ (Proposition 1.3). Clearly, $\mathcal{F}$ cannot be extended to a nowhere dense ultrafilter, as required.

To see (a) $\Longrightarrow$ (b), first use the characterization of $\mathrm{cof}(\mathcal{M})$,

$$\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$$

(see Theorem 1.3.1), and then the characterization of generic existence of P-points (see Theorem E), to appreciate the fact that we can assume $\mathrm{non}(\mathcal{M}) = \mathfrak{c}$ and $\mathfrak{d} < \mathfrak{c}$ without loss of generality. Let $\mathcal{F}$ be a filter base of size $< \mathfrak{c}$, and let $f: \omega \to 2^\omega$. It suffices to describe how we can find an infinite set $B \subseteq \omega$ such that $f[B]$ is nowhere dense and such that $\mathcal{F} \cup \{B\}$ still generates a filter. For then we can repeatedly apply this procedure working through all functions $f: \omega \to 2^\omega$ in a transfinite construction, and blow $\mathcal{F}$ up to a nowhere dense ultrafilter $\mathcal{U}$.

So let $\mathcal{F}$ and f be as in the previous paragraph. Let M be a model of set theory of size $< \mathfrak{c}$ with $f \in M$, $\mathcal{F} \subseteq M$ and such that M contains the members of a dominating family. By a delightful theorem of Bartoszyński [Ba] (see also [BJ, Lemma 2.4.8]), we know that there is a real number $g \in \omega^\omega$ such that $g(n) \notin \phi(n)$ holds for almost all n and all $\phi \in M \cap \Phi$. Here, Φ denotes the set of slaloms, i.e. $\Phi = \{\phi: \omega \to [\omega]^{<\omega}; |\phi(n)| = n \text{ for all } n\}$. Since the reals of M are dominating, there is $h \in M \cap \omega^\omega$ such that $h(n) > g(n)$ for all n. Choose (in M) a strictly increasing sequence of natural numbers $\langle k_n; n \in \omega\rangle$ with $k_0 = 0$ such that $2^{k_{n+1}-k_n} \geq h(n)$, put $I_n = [k_n, k_{n+1})$, and interpret $g(n)$ as an element of 2^{I_n}.

Next find $x \in 2^\omega$ such that $f^{-1}([x \restriction n]) \in \mathcal{F}$ for all $n \in \omega$ — note that there is without loss such an x because we may assume that $\mathcal{F}$ is an ultrafilter in the sense of M. Of course, x is unique. For each $F \in \mathcal{F}$ define a function ϕ_F with domain ω and $\phi_F(n)$ a subset of 2^{I_n} of size n recursively as follows. For each n take any $y_n^F \in 2^\omega$ with $x \restriction k_{n+1} = y_n^F \restriction k_{n+1}$ and $y_n^F \in f[F]$; guarantee that all y_n^F are distinct; and let $\phi_F(n) = \{y_i^F \restriction I_n; i < n\}$. (Note that we may assume without loss that all $f[F]$ are not nowhere dense and, a fortiori, infinite so that the choice of the y_n^F is indeed possible.) Since all this is done within M, we know that $g(n) \notin \phi_F(n)$ for almost all n and all $F \in \mathcal{F}$. Thus, if we put

$$A = \{z \in 2^\omega;\ \text{for all } n \in \omega \text{ we have either } z \restriction I_n = x \restriction I_n \text{ or } z \restriction I_n \neq g(n)\},$$

then A is closed nowhere dense (note that we must have $x \restriction I_n \neq g(n)$ for almost all n, because $x \in M$), but also $y_n^F \in A$ for almost all $n \in \omega$ and all $F \in \mathcal{F}$. This means that $f^{-1}(A) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ — and, in fact, all such intersections must be infinite because $\mathcal{F}$ contained the cofinite filter. Hence $B = f^{-1}(A)$ is as required, and the proof is complete. ∎

2.2 We can easily distill the following consequence from the preceding proof.

COROLLARY: *Every ultrafilter generated by less than* $\mathrm{cof}(\mathcal{M})$ *sets is nowhere dense.*

Proof: An old result of Ketonen [Ke] says that every ultrafilter generated by less than $\mathfrak{d}$ sets is a P-point. Hence we can assume that $\mathfrak{d} < \mathrm{non}(\mathcal{M})$ and that $\mathcal{U}$ is generated by $\mathcal{F}$ of size $< \mathrm{non}(\mathcal{M})$. Now let $f\colon \omega \to 2^\omega$. The argument of 2.1 produces $B \subseteq \omega$ such that $f[B]$ is nowhere dense and $B \cap F$ is infinite for all $F \in \mathcal{F}$. This means $B \in \mathcal{U}$, and we're done. ∎

2.3. PROOF OF THEOREM D. The argument for (b) $\Longrightarrow$ (a) is as immediate as in 2.1, and for the other direction we shall also proceed in a similar fashion. However, we shall make use of Theorem F proved in 1.5 — to see that we can assume without loss that $\mathrm{non}(\mathcal{E}) = \mathfrak{c}$ and $\mathfrak{d} < \mathfrak{c}$.

Let $\mathcal{F}$ be a filter base of size $< \mathfrak{c}$, let $f\colon \omega \to 2^\omega$ and let M be a model of set theory of size $< \mathfrak{c}$ containing f, the members of $\mathcal{F}$ as well as the elements of a dominating family. We shall produce $B \subseteq \omega$ such that $\mu(\overline{f[B]}) = 0$ and $\mathcal{F} \cup \{B\}$ still generates a filter. By the characterization of closed measure zero sets expounded in subsection 1.2, there are $h \in \omega^\omega$ and S with $S(n) \subseteq 2^{[k_n, k_{n+1})}$ and

$$\frac{|S(n)|}{2^{h(n)}} \leq \frac{1}{2^n}$$

for all n such that

$$M \cap 2^\omega \subseteq F_S := \{x \in 2^\omega; \forall^\infty n \ \ x \restriction [k_n, k_{n+1}) \in S(n)\}$$

where we put as usual $k_0 = 0$ and $k_{n+1} = k_n + h(n)$. Since the reals of M are dominating, we can assume without loss that $h \in M$.

As before find $x \in 2^\omega$ such that $f^{-1}([x \restriction n]) \in \mathcal{F}$ for all $n \in \omega$. Let $\ell_0 = 0$ and $\ell_{n+1} = \ell_n + n + 1$, choose $y_n^F \in 2^\omega \cap M$ with $x \restriction k_{\ell_n} = y_n^F \restriction k_{\ell_n}$ and $y_n^F \in f[F]$, and define $y_F \in 2^\omega$ such that $y_F \restriction [k_i, k_{i+1}) = y_j^F \restriction [k_i, k_{i+1})$ for $i = \ell_n + j$, $j \leq n$ and $n \in \omega$. We know that $y_F \in F_S$ for all $F \in \mathcal{F}$. Thus, if we put $A = \{z \in 2^\omega;$ for all $n \in \omega$ there is $j < n$ such that $z \restriction [k_i, k_{i+1}) \in S(i) \cup \{x \restriction [k_i, k_{i+1})\}$ where $i = \ell_n + j\}$, then A is closed and of measure zero, but also $y_j^F \in A$ for almost all j and all $F \in \mathcal{F}$. Thus $B = f^{-1}(A)$ is as required. ∎

2.4 We get a result analogous to 2.2.

COROLLARY: *Every ultrafilter generated by less than* $\mathrm{cof}(\mathcal{E}, \mathcal{M})$ *sets is measure zero.*

2.5. PROOF OF THEOREM E. As remarked in the introduction, the equivalence of (a) and (d) is well-known (see [Ca], [BJ, Theorem 4.4.5], [BJ1]); (d) trivially implies (b) and (c); therefore it suffices to conclude (a) from either (b) or (c). Both proofs are easy using the two parts of Theorem G. In fact, (b) $\Longrightarrow$ (a) is exactly like the corresponding direction in the two preceding proofs — and to see (c) $\Longrightarrow$ (a), simply assume $\mathfrak{d} < \mathfrak{c}$, take a model M of size $< \mathfrak{c}$ containing a dominating family and let $\mathcal{F} \in M$ be an ultrafilter which is not an ordinal ultrafilter (such ultrafilters exist in ZFC by Theorem 4.9 of [B]). Note that if $f\colon \omega \to \omega_1$ witnesses in M that $\mathcal{F}$ is not an α-ultrafilter, then it still does so in the real world because any subset of $f[\omega]$ which has order type $\leq \alpha$ is contained in such a set of M by (the "model-theoretic" version of) Theorem G. Hence $\mathcal{F}$ cannot be extended to an ordinal ultrafilter. ∎

2.6 Our results are summarized in Figure 2.

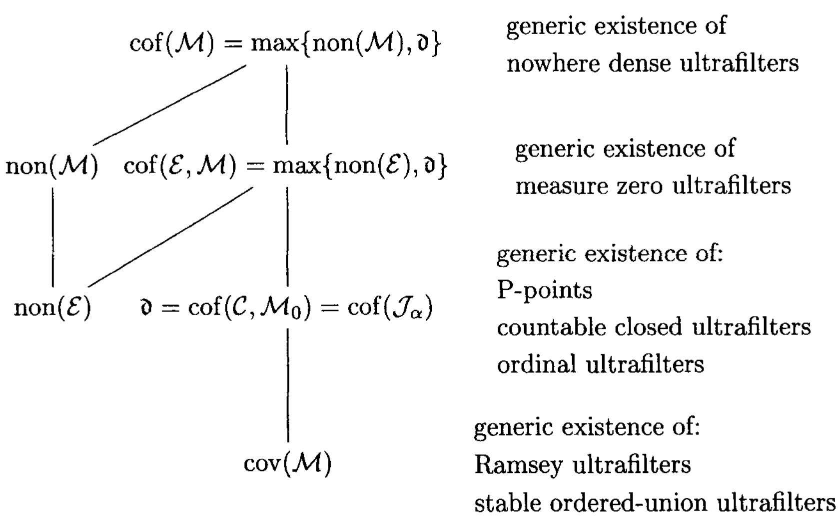

Figure 2. Summary of results.

Cardinal invariants are written in the left and middle columns and grow larger as one moves upward along the lines. In the right column, we have the class(es) of ultrafilters whose generic existence is equivalent to the statement $\mathfrak{ci} = \mathfrak{c}$ where $\mathfrak{ci}$ is the cardinal invariant immediately to the left of the ultrafilter class.

3. Consistency results

We are ready to prove our main consistency results.

3.1. Proof of Theorem A. We actually prove something slightly stronger, namely that *it is consistent to assume there are no measure zero ultrafilters yet nowhere dense ultrafilters exist generically.* $(\star)$

Given an ultrafilter $\mathcal{U}$, let $\mathbb{Q}_{\mathcal{U}}$ denote **Shelah's forcing** for adding a function $f\colon \omega \to 2^\omega$ such that for each nowhere dense set $A \subseteq 2^\omega$ in the ground model there is $U \in \mathcal{U}$ such that $f[U] \cap A = \emptyset$ (see [Sh594]). We do not need the actual definition of this forcing; it suffices to know some of its properties — namely, that it is proper and that it preserves bases for the ideal of meager sets, i.e. each meager (nowhere dense) set in the extension is contained in a meager (nowhere dense) set of the ground model. It is then immediate from (a "model-theoretic" version of) Theorem 1.3.2 that it must preserve bases for the ideal $\mathcal{E}$ as well so that every closed measure zero set (member of $\mathcal{E}$) in the extension is contained in a closed measure zero set (member of $\mathcal{E}$) of the ground model (see also Proposition 1.3 for these connections).

The following easy lemma tells us how to proceed to guarantee $(\star)$.

Lemma 1: *Assume we iterate proper forcing for ω_2 steps, use $\diamondsuit$ to go through all instances of $\mathbb{Q}_{\mathcal{U}}$, and force $\mathrm{cof}(\mathcal{M}) = \mathfrak{c} = \omega_2$ as well as $\mathrm{cof}(\mathcal{E},\mathcal{M}) = \omega_1$. Then the extension satisfies $(\star)$.*

Proof: Generic existence of nowhere dense ultrafilters follows from Theorem C. Now let $\mathcal{U}$ be any ultrafilter in the extension. We want to show that $\mathcal{U}$ is not measure zero. By $\diamondsuit$, there is an intermediate extension V_γ such that $\mathbb{Q}_\gamma = \mathbb{Q}_{\mathcal{U}\cap V_\gamma}$. This means that in $V_{\gamma+1}$ there is a function $f\colon \omega \to 2^\omega$ such that $\omega \setminus f^{-1}(N) \in \mathcal{U}\cap V_\gamma$ for each nowhere dense set $N \in V_\gamma$. Since $\mathrm{cof}(\mathcal{E}_0, \mathcal{M}_0) = \omega_1$, we can assume without loss that there is $\mathcal{F} \subseteq \mathcal{M}_0$ in V_γ which is cofinal for $\mathcal{E}_0$. Hence $\omega \setminus f^{-1}(C) \in \mathcal{U}$ for each closed measure zero set C in the final extension, as required. ∎

We see now how the *"overkill & resurrect"* method works. Shelah forcing "kills" nowhere-denseness which is then resurrected because we blow up the cardinal invariant related to generic existence of nowhere dense ultrafilters. However measure-zeroness stays dead because we keep the cardinal invariant linking nowhere dense and closed null sets small.

The additional forcing used here cofinally often in the iteration is **random forcing** $\mathbb{B}$. This means we force $\mathrm{cov}(\mathcal{N}) = \omega_2$. Since $\mathrm{cov}(\mathcal{N}) \leq \mathrm{non}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{M})$ in ZFC (see [BJ, chapter 2]), $\mathrm{cof}(\mathcal{M}) = \mathfrak{c}$ is immediate. Thus it remains to be seen that any countable support iteration of $\mathbb{Q}_{\mathcal{U}}$ and $\mathbb{B}$ keeps $\mathrm{cof}(\mathcal{E},\mathcal{M})$ small. Since $\mathrm{cof}(\mathcal{E},\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{E})\}$ by Theorem F, since both $\mathbb{B}$ and $\mathbb{Q}_{\mathcal{U}}$ are ω^ω-bounding, and since *being ω^ω-bounding* is preserved along countable support iterations, we know $\mathfrak{d} = \omega_1$ and are left with showing $\mathrm{non}(\mathcal{E}) = \omega_1$. Using a G_δ-version of Shelah's general preservation theorem (either modify [BJ, Theorem 6.1.18] accordingly or apply [E, Corollary 3.2.4] directly), it can be seen that the property *"the ground model reals do not belong to $\mathcal{E}$"* is preserved in limit steps of countable support iterations. Furthermore, since $\mathbb{Q}_{\mathcal{U}}$ preserves bases for $\mathcal{E}$ (see above), it also preserves non-$\mathcal{E}$ sets from the ground model. Hence we are left with showing

LEMMA 2: *Assume $Y \notin \mathcal{E}$. Then $Y \notin \mathcal{E}$ after adding a random real.*

Proof: This is a standard probabilistic argument. Let $A \in \mathcal{E}$ in the random extension. By 1.2 we can assume $A = F_S$ for some $S \in \mathcal{C}_h$, $h \in \omega^\omega$, i.e.

$$A = F_S = \{x \in 2^\omega; \forall^\infty n \;\; x \restriction [k_n, k_{n+1}) \in S(n)\}$$

where $k_0 = 0, \ldots, k_{n+1} = k_n + h(n)$. Since $\mathbb{B}$ is ω^ω-bounding, we can assume that h lies in the ground model. Let $\dot{S}$ be a $\mathbb{B}$-name for S. For $s \in 2^{[k_n,k_{n+1})}$ let $b_s = [\![s \in \dot{S}(n)]\!] \in \mathbb{B}$. Since

$$1 = [\![\frac{|\dot{S}(n)|}{2^{h(n)}} \leq \frac{1}{2^n}]\!]$$

we know by Fubini that

$$\sum_{s\in 2^{[k_n,k_{n+1})}} \mu(b_s) = \int |\dot{S}(n)| \leq \frac{2^{h(n)}}{2^n} = 2^{h(n)-n}.$$

Let

$$T(n) = \{s \in 2^{[k_n,k_{n+1})}; \mu(b_s) \geq 1/2^{n/2}\}.$$

Thus

$$\frac{|T(n)|}{2^{h(n)}} \leq \frac{1}{2^{n/2}}.$$

Clearly

$$F_T = \{x \in 2^\omega; \forall^\infty n \;\; x \restriction [k_n, k_{n+1}) \in T(n)\}$$

is still a member of $\mathcal{E}$. Hence there is $y \in Y \setminus F_T$. Let us now argue that $[\![y \notin F_{\dot{S}}]\!] = 1$. Given any $b \in \mathbb{B}$ find $I \subseteq \omega$ infinite such that $y \restriction [k_i, k_{i+1}) \notin T(i)$ for $i \in I$ and such that

$$\sum_{i\in I} \mu(b_{y\restriction[k_i,k_{i+1})}) < \mu(b).$$

This is clearly possible by definition of T. Now put $c = b \setminus \bigcup_{i\in I} b_{y\restriction[k_i,k_{i+1})}$. Then $c \Vdash y \notin F_{\dot S}$, as required. ∎

This completes the proof of Theorem A. ∎

3.2. Proof of Theorem B. Again, we prove something slightly stronger, namely that *it is consistent to assume there are no countable closed ultrafilters yet measure zero ultrafilters exist generically.* $(\star\star)$

The following lemma is the analogue of Lemma 3.1.1.

Lemma 1: *Assume we iterate ω^ω-bounding proper forcing for ω_2 steps, use $\diamondsuit$ to go through all instances of Shelah's forcing $\mathbb{Q}_{\mathcal{U}}$, and force $\mathrm{cof}(\mathcal{E},\mathcal{M}) = \mathfrak{c} = \omega_2$. Then the extension satisfies $(\star\star)$.*

Proof: Generic existence of measure zero ultrafilters follows from Theorem D. To show that no ultrafilter is countable closed, we argue as in Lemma 3.1.1, using Theorem G ($\mathrm{cof}(\mathcal{C},\mathcal{M}_0) = \mathfrak{d} = \omega_1$) instead. ∎

This Lemma clearly reduces our task to proving (by the easy part of Theorem F, the inequality $\mathrm{cof}(\mathcal{E},\mathcal{M}) \geq \mathrm{non}(\mathcal{E})$)

Lemma 2: *There is an ω^ω-bounding proper forcing notion $\mathbb{E}$ which adds $S \in \mathcal{C}_h$ for some $h \in \omega^\omega$ such that all ground model reals are contained in $F_S = \{x; \forall^\infty n\ x \restriction [k_n, k_{n+1}) \in S(n)\} \in \mathcal{E}$.*

Proof: The forcing we are going to define belongs to a class of forcing notions introduced by Shelah in [Sh326] (see also [BJ, 7.3.B]). We therefore give only a brief outline of the argument. Details can be found in the references.

Let $h \in \omega^\omega$ be a function growing fast enough. We want to add S such that $S(n) \subseteq 2^{h(n)}$ and

$$\frac{|S(n)|}{2^{h(n)}} \leq \frac{1}{2^n}.$$

Hence let $\mathbb{E}$ consist of all trees T with

(a) $s \in T$ iff $s(n) \subseteq 2^{h(n)}$ and $|s(n)|/2^{h(n)} \leq 1/2^n$ for all $n < |s|$

(b) there is $f \in \omega^\omega$ increasing with $\lim_{n\to\infty} f(n) = \infty$ such that for $s \in T$ with $stem(T) \subseteq s$ we have that

$$succ_T(s) := \{\sigma \subseteq 2^{h(|s|)}; |\sigma|/2^{h(|s|)} \leq 1/2^{|s|} \text{ and } s\hat{\ }\sigma \in T\}$$

has norm at least $f(|s|)$.

Here we define the **norm** $\nu(A)$ of a set

$$A \subseteq \Sigma_n := \left\{\sigma \subseteq 2^{h(n)}; \frac{|\sigma|}{2^{h(n)}} \leq \frac{1}{2^n}\right\}$$

recursively as follows:

(A) $\nu(A) \geq 0$ iff $\bigcup A = 2^{h(n)}$,

(B) $\nu(A) \geq \ell + 1$ iff whenever $B \cup C = A$ then either $\nu(B) \geq \ell$ or $\nu(C) \geq \ell$.

As a consequence of (A) and (B) we get

(C) whenever $\nu(A) \geq \ell + 1$ and $\tau \in 2^{h(n)}$ then $\nu(A_\tau) \geq \ell$ where $A_\tau = \{\sigma \in A; \tau \in \sigma\}$.

It is easy to see that as long as h is growing fast enough, $\lim_{n\to\infty} \nu(\Sigma_n) = \infty$, so that our forcing is well-defined. Order $\mathbb{E}$ by inclusion. Standard arguments show that $\mathbb{E}$ is ω^ω-bounding and proper (this uses (B) in the definition of the norm, see [Sh326], [BJ, 7.3.B] for similar arguments). Finally, (C) and an easy genericity argument yield that $f \restriction [k_n, k_{n+1}) \in S(n)$ for almost all $n \in \omega$ and all $f \in 2^\omega$ in the ground model — where, as usual, $k_0 = 0$ and $k_{n+1} = k_n + h(n)$. Here, $S \in \mathcal{C}_h$ denotes the generic branch (i.e. the unique branch belonging to all trees of the generic filter). ∎

This concludes the proof of Theorem B. ∎

4. Further results and questions

4.1 A natural question to address about Figure 1 in the Introduction concerning the inclusion relations between various ultrafilter classes is whether any of the arrows can be reversed in ZFC. In most instances, this has been answered by Baumgartner [B, Theorems 1.4, 4.3 and 4.13] by exhibiting an ultrafilter belonging to one class without belonging to the other under MA. We consider the remaining three cases — the fact that new cases arise is mostly due to our introduction of countable closed ultrafilters.

THEOREM 1 ($MA(\sigma$-centered)): *There is a discrete ultrafilter which is not measure zero.*

THEOREM 2 ($\mathfrak{d} = \mathfrak{c}$): *There is a countable closed ultrafilter which is not discrete.*

We have no information about the third case, but it seems plausible that the following can be shown.

CONJECTURE (CH): *There is a countable closed ultrafilter which is not an ordinal ultrafilter.*

This may only be a matter of detail. See [B, Theorem 4.13] for a result which might be related.

4.2. Proof of Theorem 4.1.1. The required ultrafilter can be easily built up recursively once the following result which strengthens Lemma 1.6 in [B] has been established.

Lemma (MA(σ-centered)): *Given $f: 2^{<\omega} \to 2^\omega$ and a filter base $\mathcal{F} \subseteq [2^{<\omega}]^\omega$ of size $< \mathfrak{c}$ such that $\mu(\bar{A}) > 0$ for all $A \in \mathcal{F}$, there is $B \in [2^{<\omega}]^\omega$ such that $f[B]$ is discrete and $\mu(\overline{A \cap B}) > 0$ for all $A \in \mathcal{F}$.*

Proof: In case there is $x \in 2^\omega$ with $\mu(\overline{A \cap f^{-1}(x)}) > 0$ for all $A \in \mathcal{F}$, there is nothing to show. So assume this is not the case.

Suppose first that for all $A \in \mathcal{F}$ there is $y \in f[2^{<\omega}]$ with $\mu(\overline{A \cap f^{-1}(y)}) > 0$. If, in addition, there is $x \in 2^\omega$ such that for all k and all $A \in \mathcal{F}$ there is such $y \in [x \restriction k]$, then a real Cohen over $\mathcal{F}, f, x$ easily induces a sequence

$$\{y_n \in f[2^{<\omega}]; n \in \omega\}$$

which converges to x (and thus is discrete) and for all $A \in \mathcal{F}$ there is $n \in \omega$ with $\mu(\overline{A \cap f^{-1}(y_n)}) > 0$, and we're done. In case there is no such x, for each $x \in 2^\omega$ we can find k_x such that, for some $A \in \mathcal{F}$, there is no such $y \in [x \restriction k_x]$. Then construe Cohen forcing as forcing with finite functions $\phi: f[2^{<\omega}] \to \omega$ such that if $x, y \in dom(\phi)$, then $\phi(x) \geq k_x$ and $[x \restriction \phi(x)] \cap [y \restriction \phi(y)] = \emptyset$. The domain of the generic function Φ_G (over $\mathcal{F}, f$) will be discrete, and a straightforward genericity argument shows that for all $A \in \mathcal{F}$ there is $y \in dom(\Phi_G)$ with $\mu(\overline{A \cap f^{-1}(y)}) > 0$.

All of this shows that the interesting case occurs when there is $A \in \mathcal{F}$ such that $\mu(\overline{A \cap f^{-1}(y)}) = 0$ for all $y \in f[2^{<\omega}]$. Working below such A, if necessary, we can assume without loss that $A = 2^{<\omega}$. Let $\mathbb{P}$ consist of pairs (ϕ, Φ) where $\phi: f[2^{<\omega}] \to \omega$ and $\Phi: \mathcal{F} \to \mathbb{Q}^+$ are finite partial functions such that

(i) $x \neq y \in dom(\phi)$ implies that $[x \restriction \phi(x)] \cap [y \restriction \phi(y)] = \emptyset$,

(ii) $\mu(\overline{A \setminus \bigcup_{x \in dom(\phi)} f^{-1}[x \restriction \phi(x)]}) > \Phi(A)$ for all $A \in dom(\Phi)$,

(iii) $\mu(\overline{A \setminus \bigcup_{x \in dom(\phi)} f^{-1}[x \restriction \phi(x)]}) > 0$ for all $A \in \mathcal{F}$.

Put $(\phi, \Phi) \leq (\psi, \Psi)$ iff $\phi \supseteq \psi$, $dom(\Phi) \supseteq dom(\Psi)$ and $\Phi(A) \geq \Psi(A)$ for all $A \in dom(\Psi)$. Clearly $\mathbb{P}$ is σ-centered. Also if G is a filter on $\mathbb{P}$, then the domain of the generic function $C_G = \bigcup\{dom(\phi);$ there is Φ with $(\phi, \Phi) \in G\} \subseteq 2^{<\omega}$ is discrete. Hence it suffices to exhibit, for each $A \in \mathcal{F}$, a countable family $\mathcal{D}_A$ of dense sets such that if G meets enough of them, then $\mu(\overline{A \cap B}) > 0$ where $B = f^{-1}(C_G)$.

Fix $A \in \mathcal{F}$. Given $\epsilon \in \mathbb{Q}^+$, let $D_{A,\epsilon} = \{(\phi, \Phi) \in \mathbb{P}; A \in dom(\Phi)$ and $\Phi(A) \geq \epsilon\}$. By (iii), $D_A = \bigcup_{\epsilon \in \mathbb{Q}^+} D_{A,\epsilon}$ is dense in $\mathbb{P}$. Now fix $\epsilon \in \mathbb{Q}^+$. Let T_ϵ be the set of all finite subets t of $2^{<\omega}$ such that $\mu([t]) < \epsilon$ where $[t] \subseteq 2^\omega$ denotes the (closed) set of functions which extend some $\sigma \in t$, i.e. $[t] = \bigcup_{\sigma \in t}[\sigma]$. For such t, put $D_{A,\epsilon,t} = \{(\phi, \Phi) \in \mathbb{P}$; there is $x \in dom(\phi)$ such that $f^{-1}(x) \cap (A \setminus [t]) \neq \emptyset\}$. We claim that $D_{A,\epsilon,t}$ is dense below each element of $D_{A,\epsilon}$.

To see this take $(\phi, \Phi) \in D_{A,\epsilon}$. Let $n = |\Phi|$. For $F \in \mathcal{F}$, put $F' = F \setminus \bigcup_{x \in dom(\phi)} f^{-1}[x \restriction \phi(x)]$. Let m be such that

$$\mu(\bar{F}') > \frac{(m+1) \cdot \Phi(F)}{m}$$

for all $F \in dom(\Phi)$. By (ii) we know that $A' \not\subseteq [t]$, in fact $\mu(\overline{A' \setminus [t]}) > 0$, and hence we can find distinct x_i, $i \in nm+2$, with $f^{-1}(x_i) \cap (A' \setminus [t]) \neq \emptyset$ (recall that we assumed at the beginning that all fibers $f^{-1}(x)$ have null closure). Next find k_i, $i < nm+2$, such that the $[x_i \restriction k_i]$ are pairwise disjoint and such that they are disjoint from the $[x \restriction \phi(x)]$ for $x \in dom(\phi)$ (this is possible by construction). Given $F \in dom(\Phi)$, call i bad for F if

$$\mu(\overline{F' \cap f^{-1}[x_i \restriction k_i]}) > \frac{\Phi(F)}{m}.$$

Then, for each $F \in dom(\Phi)$, select m elements from $nm+2$ such that either all are bad for F or the bad ones are contained among the chosen ones. There are at least two elements of $nm+2$ left. Choose one of them — say j — such that, if we put $\psi = \phi \cup \{\langle x_j, k_j \rangle\}$ and $\Psi = \Phi$, (iii) is still satisfied for (ψ, Ψ). This is clearly possible for there can be at most one i such that $\mu(\overline{F' \setminus f^{-1}[x_i \restriction k_i]}) = 0$ for some $F \in \mathcal{F}$ because $\mathcal{F}$ is a filter. We now argue that (ii) holds as well: fix $F \in dom(\Phi)$. Then either j is not bad for F in which case

$$\mu(\overline{F' \cap f^{-1}[x_j \restriction k_j]}) \leq \frac{\Phi(F)}{m}$$

and thus

$$\mu(\overline{F' \setminus f^{-1}[x_j \restriction k_j]}) \geq \mu(\bar{F}') - \frac{\Phi(F)}{m} > \Phi(F),$$

or all chosen elements $i_0, \ldots, i_{m-1}$ were bad and thus

$$\mu(\overline{F' \setminus f^{-1}[x_j \restriction k_j]}) \geq \mu(\overline{F' \cap \bigcup_{r<m} f^{-1}[x_{i_r} \restriction k_{i_r}]}) > m \cdot \frac{\Phi(F)}{m} = \Phi(F).$$

So we're done in both cases and found $(\psi, \Psi) \leq (\phi, \Phi)$ in $D_{A,\epsilon,t}$.

Thus we can require that G meets some $D_{A,\epsilon}$ and then all $D_{A,\epsilon,t}$ for $t \in T_\epsilon$, for all $A \in \mathcal{F}$. We are left with checking that $\mu(\overline{A \cap B}) > 0$ for all $A \in \mathcal{F}$. Fix $A \in \mathcal{F}$. Let ϵ be such that $D_{A,\epsilon} \cap G \neq \emptyset$. If $\overline{A \cap B}$ were null, we could find a finite $t \subseteq 2^{<\omega}$ with $\mu([t]) < \epsilon$ and $\overline{A \cap B} \subseteq [t]$. However, $D_{A,\epsilon,t} \cap G \neq \emptyset$. Thus there is $x \in C_G$ with $f^{-1}(x) \cap (A \setminus [t]) \neq \emptyset$, i.e. $B \cap (A \setminus [t]) \neq \emptyset$, a contradiction.

∎

4.3. PROOF OF THEOREM 4.1.2. Baumgartner [B, Theorem 2.6] proved that if $\mathcal{U}$ is a non-principal ultrafilter on ω, then $\mathcal{U}^\omega$ is not discrete, where $\mathcal{U}^1 = \mathcal{U}$, $\mathcal{U}^{n+1} = \mathcal{U} \times \mathcal{U}^n$ and $\mathcal{U}^\omega = \sum_{\mathcal{U}} \mathcal{U}^n$. Here, given ultrafilters $\mathcal{U}$ and $\mathcal{V}_n$, $n \in \omega$, on ω, we define $\mathcal{W} = \sum_{\mathcal{U}} \mathcal{V}_n$ on $\omega \times \omega$ by

$$A \in \mathcal{W} \iff \{n; \{m; (n,m) \in A\} \in \mathcal{V}_n\} \in \mathcal{U}$$

and write $\mathcal{U} \times \mathcal{V}$ for $\sum_{\mathcal{U}} \mathcal{V}$. Hence it suffices to prove

PROPOSITION: *If $\mathcal{U}$ is a P-point, then $\mathcal{U}^\omega$ is countable closed.*

Proof: Think of $\mathcal{U}^\omega$ as an ultrafilter on $X = \bigcup_n \{n\} \times \omega^n$. Let $f\colon X \to 2^\omega$ be given. Find $x \in 2^\omega$ such that $f^{-1}([x \restriction k]) \in \mathcal{U}^\omega$ for all k. For each n, find $x_n \in 2^\omega$ such that $\{m; (n,m) \in f^{-1}([x_n \restriction k])\} \in \mathcal{U}^n$ for all k. Using that $\mathcal{U}$ is P-point, find $A \in \mathcal{U}$ such that $\{x_n; n \in A\}$ is either the single point x or a sequence converging to x. This argument may be repeated for each $\mathcal{U}^n$ (instead of $\mathcal{U}^\omega$), finding sequences converging to x_n, etc. Thus we construct a set $B \in \mathcal{U}^\omega$ such that $\overline{f[B]}$ is a countable set with Cantor–Bendixson rank at most $\omega + 1$. ∎

4.4. PROBLEMS. There are two main problems left untouched in our work, and whose solution should shed new light on our ultrafilter classes.

(1) *Characterize generic existence of discrete and of scattered ultrafilters!*

The only thing known is that $\mathrm{cof}(\mathcal{M}) = \mathfrak{c}$ (or, equivalently, generic existence of nowhere dense ultrafilters) is not sufficient for getting generic existence of scattered ultrafilters [B, Theorem 3.5]. This means that the approach taken for measure zero and countable closed ultrafilters cannot work for it is well-known that $\mathrm{cof}(\mathcal{D}, \mathcal{M}_0) = \mathrm{cof}(\mathcal{M})$ where $\mathcal{D}$ is the family of discrete sets (simply note that given any N closed nowhere dense, we can construct D discrete such that $\bar{D} = D \cup N$).

(2) *Is it consistent there are countable closed (ordinal) ultrafilters and no P-points?*

Since generic existence of these three classes of ultrafilters is equivalent (Theorem E), the crude "overkill & resurrect" method of Theorems A and B cannot work, and finer techniques would be needed. Of course, one may ask similar questions about discrete and scattered ultrafilters.

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